0. Title of presentation

Modeling of coupled rigid-elastic multibody systems using DAE and ODE formalisms

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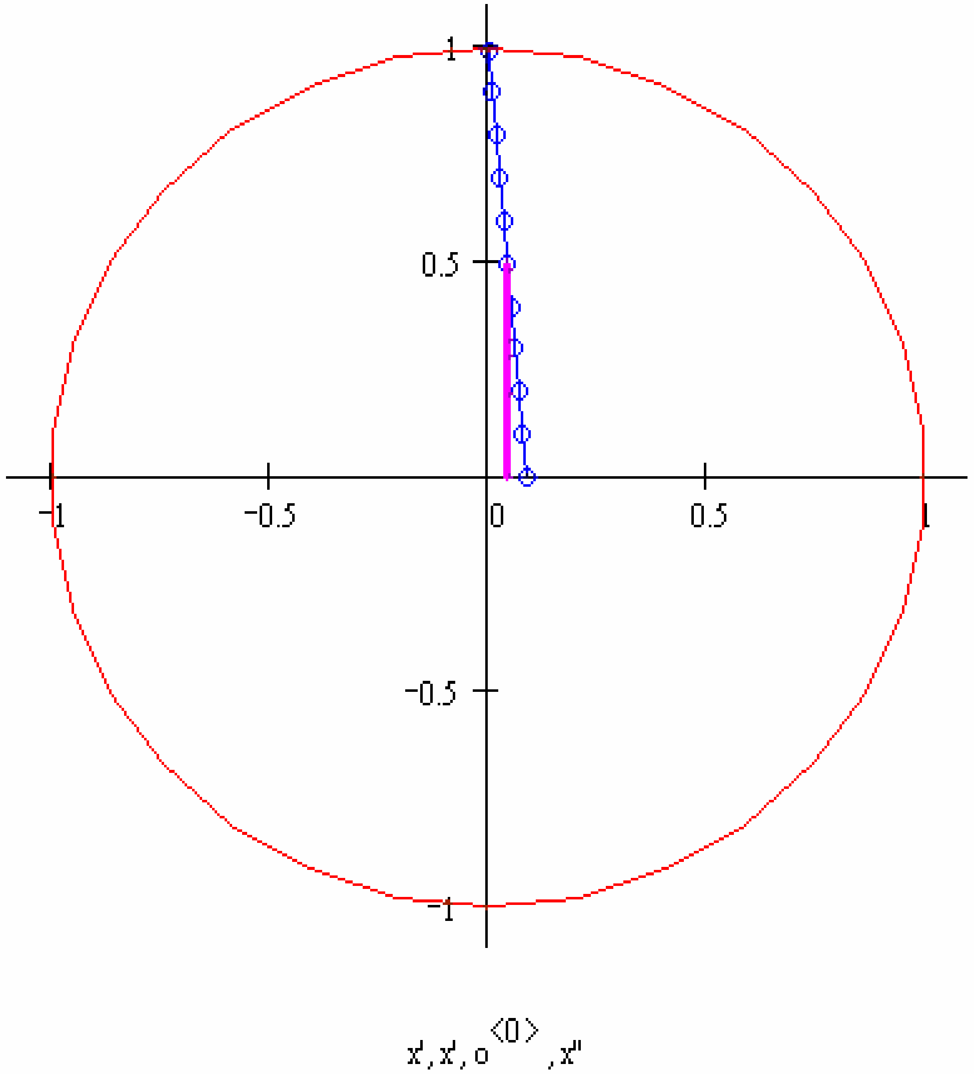
0.1. Example of coupled rigid-elastic system



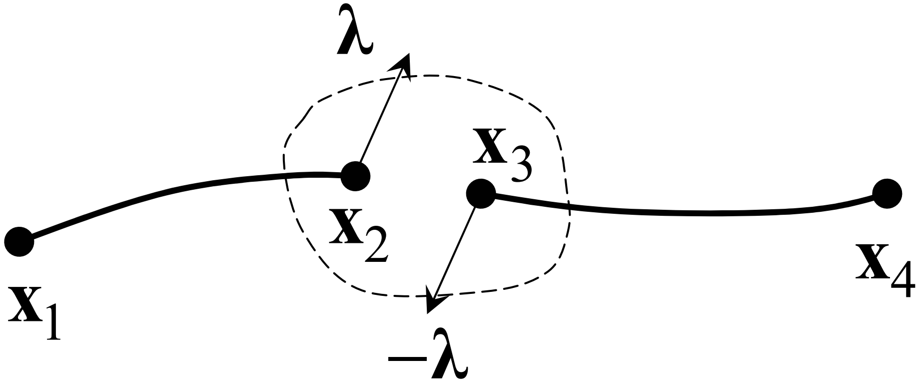
The most common way of modeling such a system employs DAE

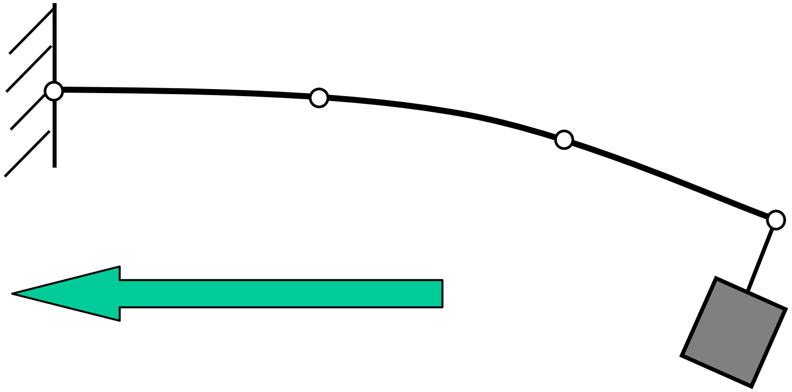
$$\begin{cases} \mathbf{M}_1\ddot{\mathbf{x}}_1 = \mathbf{f}_1 + (\mathbf{G}_1^{\mathrm{T}}\boldsymbol{\lambda}) \\ \mathbf{M}_2\ddot{\mathbf{x}}_2 = \mathbf{f}_2 + (\mathbf{G}_2^{\mathrm{T}}\boldsymbol{\lambda}) \\ (\mathbf{g}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0}) \end{cases}$$

$$\mathbf{G}_k = \frac{\partial \mathbf{g}}{\partial \mathbf{x}_k^{\mathrm{T}}}$$

However in many cases it is possible to avoid DAE, e.g. in Finite Element Method

0.2. The Assembling Procedure in FEM





$$\begin{cases} \mathbf{M}_{11}\ddot{\mathbf{x}}_1 + \mathbf{M}_{12}\ddot{\mathbf{x}}_2 = \mathbf{f}_1 \\ \mathbf{M}_{21}\ddot{\mathbf{x}}_1 + \mathbf{M}_{22}\ddot{\mathbf{x}}_2 = \mathbf{f}_2 + (\boldsymbol{\lambda}) \end{cases}$$

$$\begin{cases} \mathbf{M}_{33}\ddot{\mathbf{x}}_3 + \mathbf{M}_{34}\ddot{\mathbf{x}}_4 = \mathbf{f}_3 - (\boldsymbol{\lambda}) \\ \mathbf{M}_{43}\ddot{\mathbf{x}}_3 + \mathbf{M}_{44}\ddot{\mathbf{x}}_4 = \mathbf{f}_4 \end{cases}$$

Constraints are trivial:

$$(\mathbf{x}_2 \equiv \mathbf{x}_3 = \mathbf{x}_*)$$

$$\left[\begin{array}{c|c|c} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{O} \\ \hline \mathbf{M}_{21} & \mathbf{M}_{22} + \mathbf{M}_{33} & \mathbf{M}_{34} \\ \hline \mathbf{O} & \mathbf{M}_{43} & \mathbf{M}_{44} \end{array}\right] \left\{\begin{array}{c} \ddot{\mathbf{x}}_1 \\ \ddot{\mathbf{x}}_* \\ \ddot{\mathbf{x}}_4 \end{array}\right\} = \left\{\begin{array}{c} \mathbf{f}_1 \\ \mathbf{f}_2 + \mathbf{f}_3 \\ \mathbf{f}_4 \end{array}\right\}$$

0.3. Generalized coordinates of a rigid body

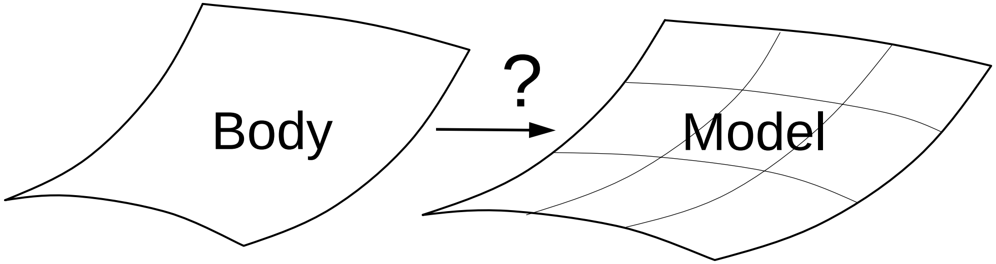
Translational motion of rigid bodies is usually simulated with Cartesian coordinates.

To describe the **rotational motion** of the body we use

rotation angle φ in 2D case
or any triplet α_1, α_2, α_3 of angles in 3D case.

(We do not consider the Euler-Rodriguez parameters)

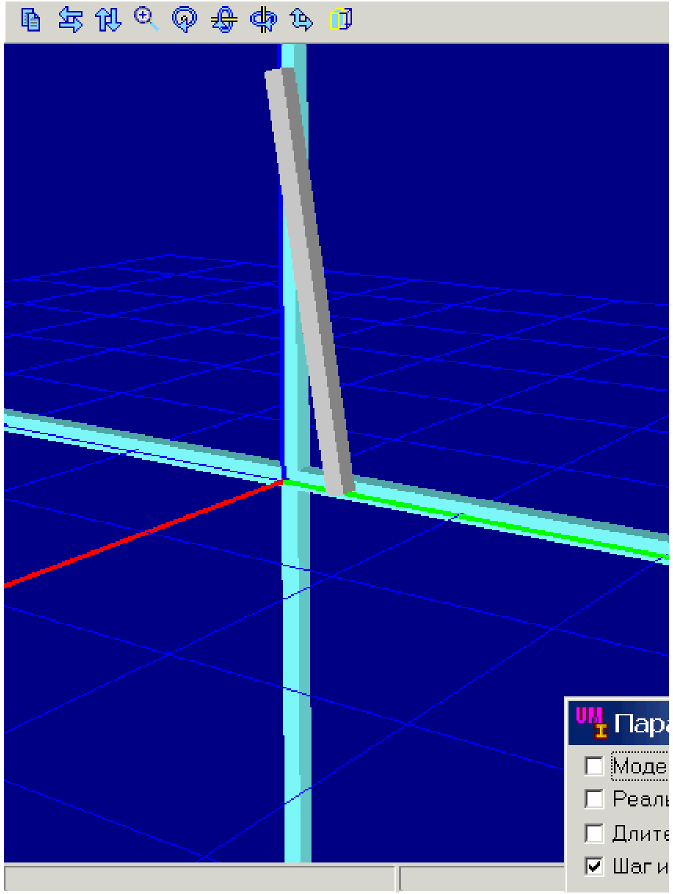
0.4. Finite-element formulations for elastic bodies



Consider two large-displacement approaches for simulation of elastic bodies:

1. Large rotation vector formulation (*J. C. Simo, 1985*)
2. Absolute nodal coordinate formulation (*A. Shabana, 1996*)

Example of large-displacement elastic system



Simulation tool is
Universal Mechanism
by Prof. *D. Pogorelov*

1. Large rotation vector formulation by *J. Simo*

1.1. Large rotation vector formulation for 2D beam

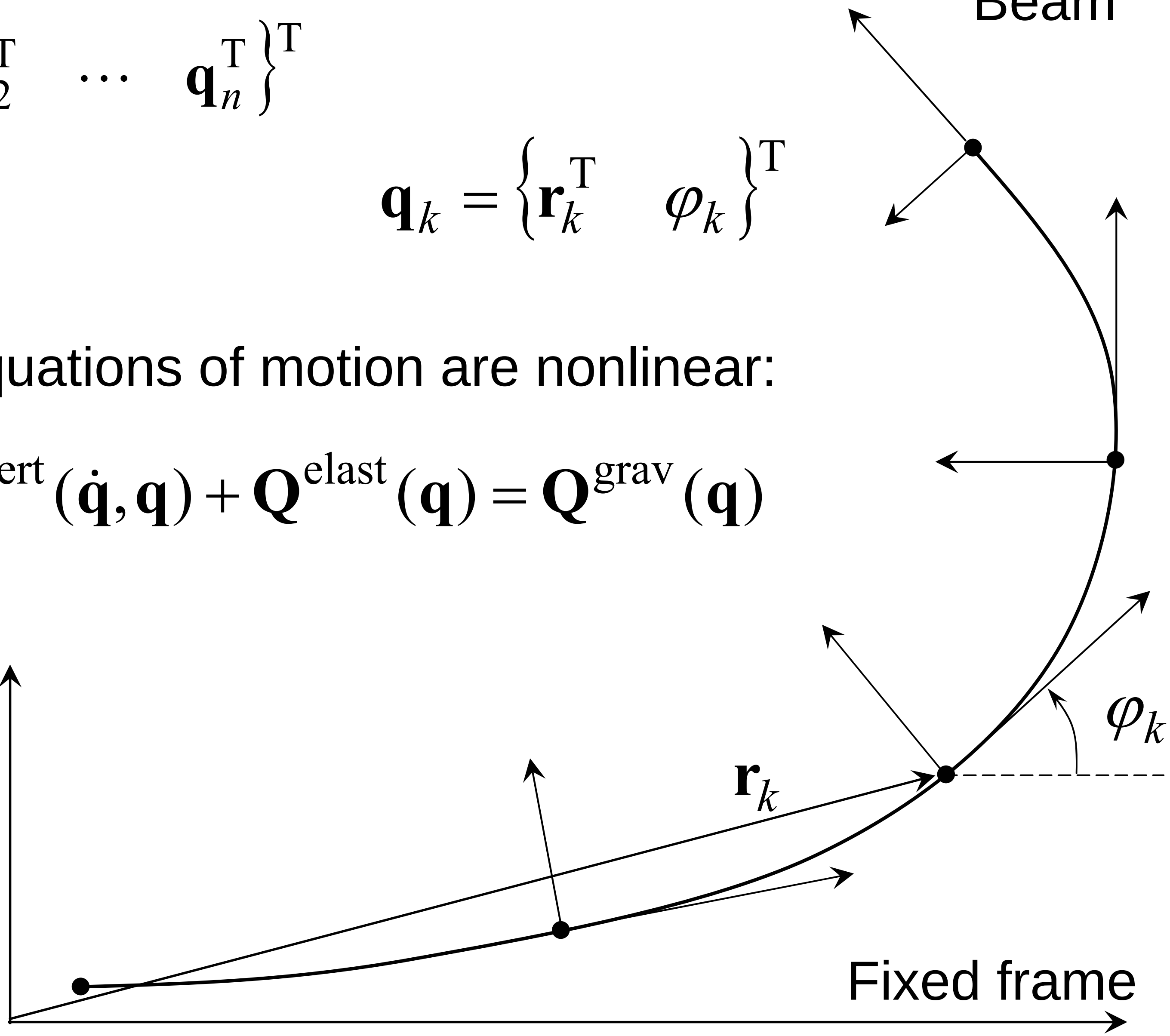
Vector of nodal coordinates

$$\mathbf{q} = \left\{ \mathbf{q}_1^{\mathrm{T}} \quad \mathbf{q}_2^{\mathrm{T}} \quad \cdots \quad \mathbf{q}_n^{\mathrm{T}} \right\}^{\mathrm{T}}$$

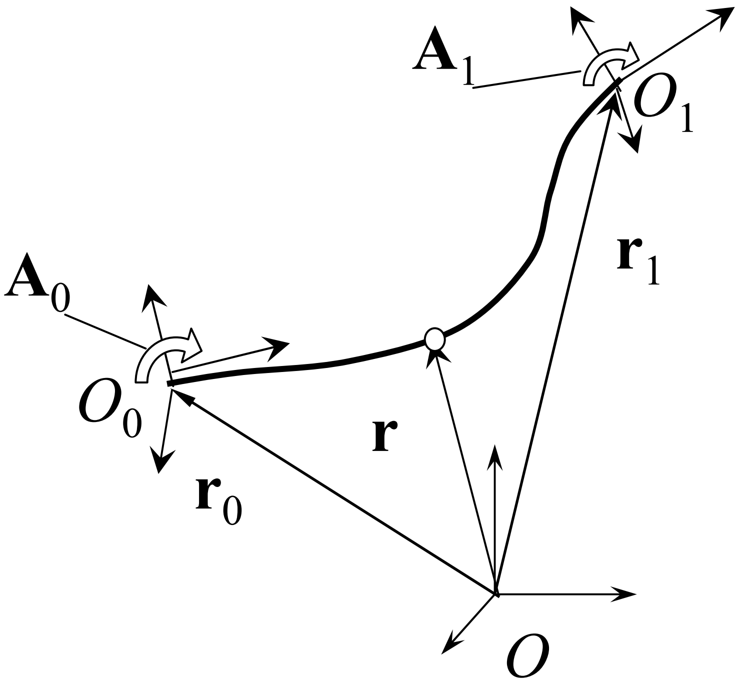
$$\mathbf{q}_k = \left\{ \mathbf{r}_k^{\mathrm{T}} \quad \varphi_k \right\}^{\mathrm{T}}$$

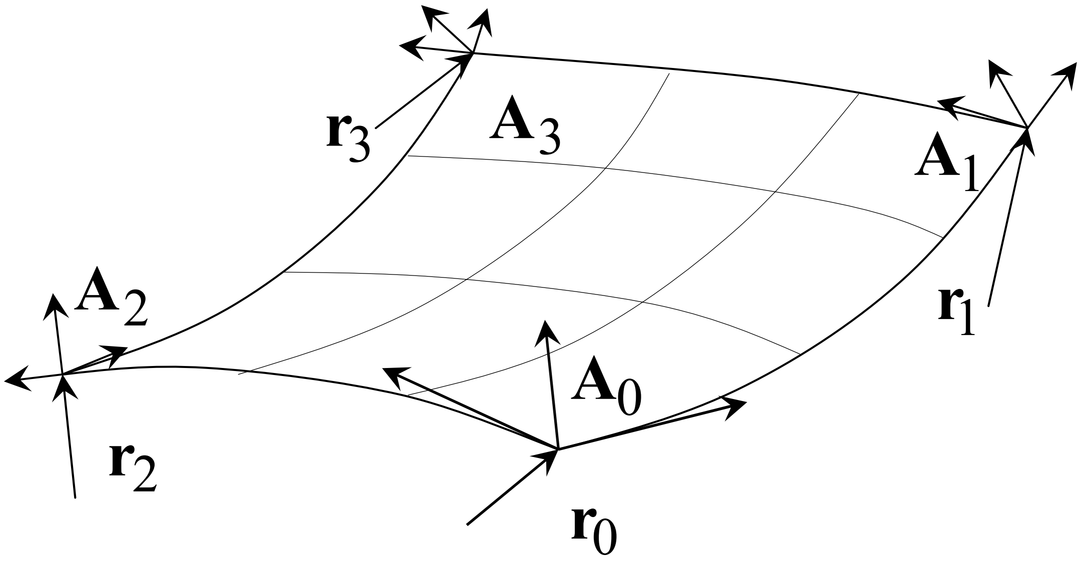
All terms of equations of motion are nonlinear:

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{Q}^{\mathrm{inert}}(\dot{\mathbf{q}},\mathbf{q}) + \mathbf{Q}^{\mathrm{elast}}(\mathbf{q}) = \mathbf{Q}^{\mathrm{grav}}(\mathbf{q})$$



1.2. 3D beam and plate FEs using large rotations





Coordinates

$$\mathbf{q} = \left\{ \mathbf{q}_0^{\mathrm{T}} \quad \mathbf{q}_1^{\mathrm{T}} \right\}^{\mathrm{T}}$$

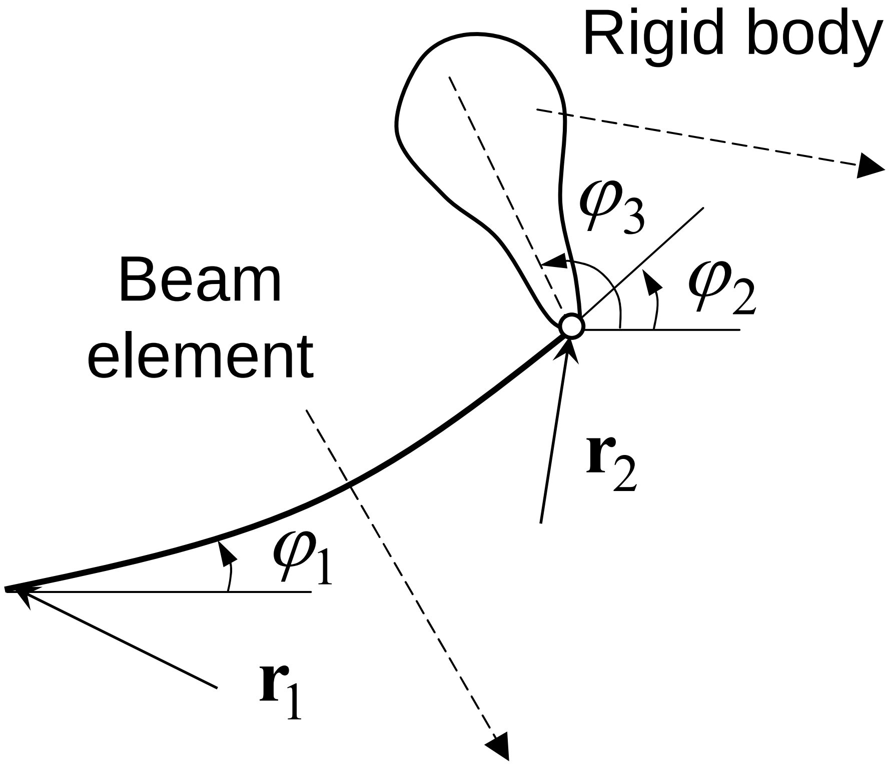
$$\mathbf{q} = \left\{ \mathbf{q}_0^{\mathrm{T}} \quad \mathbf{q}_1^{\mathrm{T}} \quad \mathbf{q}_2^{\mathrm{T}} \quad \mathbf{q}_3^{\mathrm{T}} \right\}^{\mathrm{T}}$$

$$\mathbf{q}_k = \begin{Bmatrix} \mathbf{r}_k \\ \boldsymbol{\varphi}(\mathbf{A}_k) \end{Bmatrix}$$

A triplet of orientation angles

$$\boldsymbol{\varphi}(\mathbf{A}_k) = \left\{ \alpha_{1k} \quad \alpha_{2k} \quad \alpha_{3k} \right\}^{\mathrm{T}}$$

1.3. Revolute joint between beam and rigid body



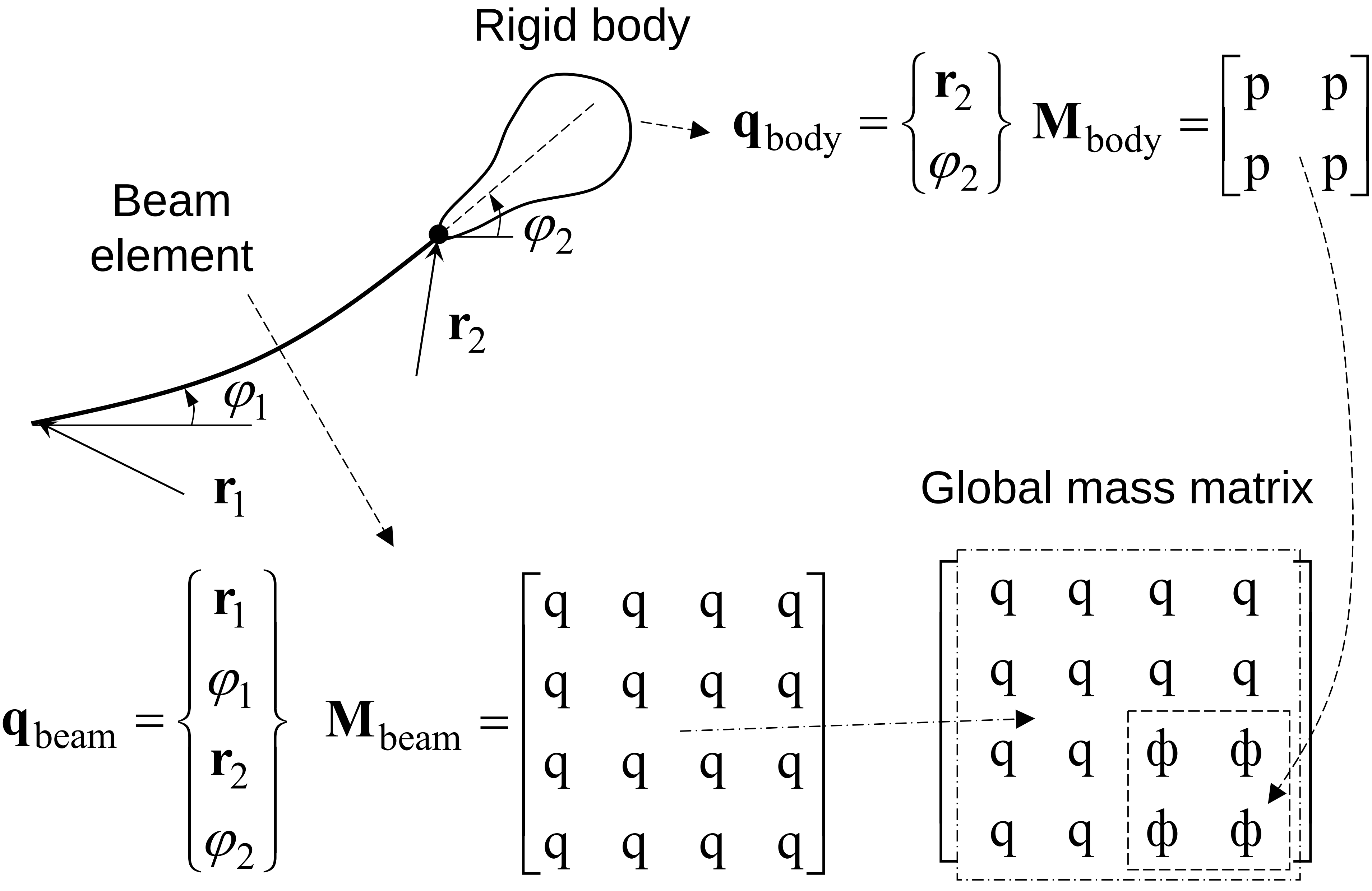
$$\mathbf{q}_{\text{body}} = \begin{Bmatrix} \mathbf{r}_2 \\ \varphi_3 \end{Bmatrix} \quad \mathbf{M}_{\text{body}} = \begin{bmatrix} p & p \\ p & p \end{bmatrix}$$

$$\mathbf{q}_{\text{beam}} = \begin{Bmatrix} \mathbf{r}_1 \\ \varphi_1 \\ \mathbf{r}_2 \\ \varphi_2 \end{Bmatrix} \quad \mathbf{M}_{\text{beam}} = \begin{bmatrix} q & q & q & q \\ q & q & q & q \\ q & q & q & q \\ q & q & q & q \end{bmatrix}$$

Global mass matrix

$$\begin{bmatrix} q & q & q & q & \\ q & q & q & q & \\ q & q & \phi & q & p \\ q & q & q & q & \\ & & p & & p \end{bmatrix}$$

1.4. Fixed connection between beam and rigid body



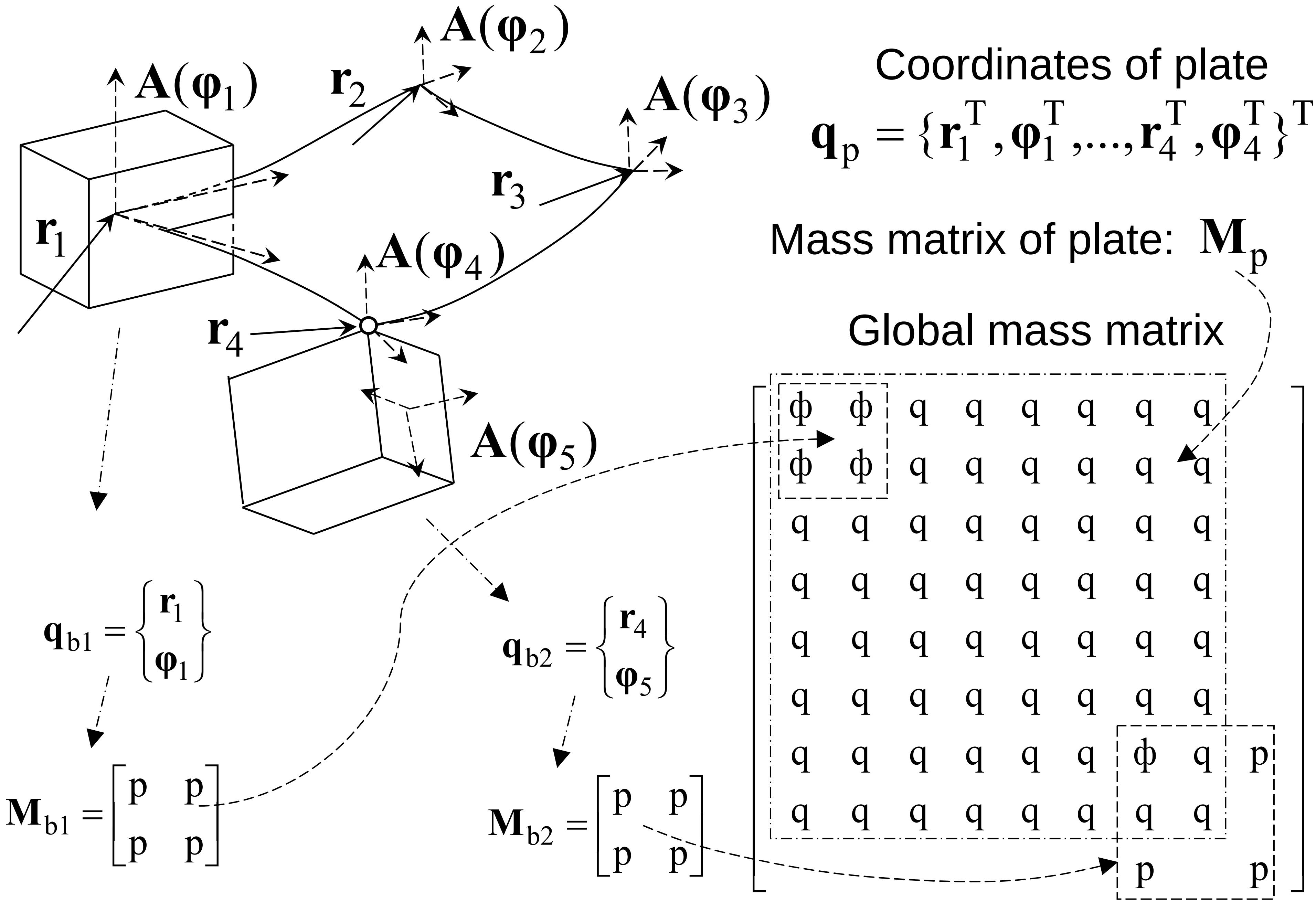
$$\mathbf{q}_{\text{body}} = \begin{Bmatrix} \mathbf{r}_2 \\ \varphi_2 \end{Bmatrix} \quad \mathbf{M}_{\text{body}} = \begin{bmatrix} \text{p} & \text{p} \\ \text{p} & \text{p} \end{bmatrix}$$

$$\mathbf{q}_{\text{beam}} = \begin{Bmatrix} \mathbf{r}_1 \\ \varphi_1 \\ \mathbf{r}_2 \\ \varphi_2 \end{Bmatrix} \quad \mathbf{M}_{\text{beam}} = \begin{bmatrix} \text{q} & \text{q} & \text{q} & \text{q} \\ \text{q} & \text{q} & \text{q} & \text{q} \\ \text{q} & \text{q} & \text{q} & \text{q} \\ \text{q} & \text{q} & \text{q} & \text{q} \end{bmatrix}$$

Global mass matrix

$$\begin{bmatrix} \text{q} & \text{q} & \text{q} & \text{q} \\ \text{q} & \text{q} & \text{q} & \text{q} \\ \text{q} & \text{q} & \phi & \phi \\ \text{q} & \text{q} & \phi & \phi \end{bmatrix}$$

1.5. Attaching rigid bodies to a plate



Coordinates of plate

$$\mathbf{q}_p = \{\mathbf{r}_1^T, \boldsymbol{\varphi}_1^T, \ldots, \mathbf{r}_4^T, \boldsymbol{\varphi}_4^T\}^T$$

Mass matrix of plate: $\mathbf{M}_p$

Global mass matrix

$$\left[\begin{array}{ccccccccc} ф & ф & q & q & q & q & q & q & \\ ф & ф & q & q & q & q & q & q & \\ q & q & q & q & q & q & q & q & \\ q & q & q & q & q & q & q & q & \\ q & q & q & q & q & q & q & q & \\ q & q & q & q & q & q & q & q & \\ q & q & q & q & q & q & ф & q & p \\ q & q & q & q & q & q & q & q & \\ & & & & & & p & & p \end{array}\right]$$

$$\mathbf{q}_{b1} = \begin{Bmatrix} \mathbf{r}_1 \\ \boldsymbol{\varphi}_1 \end{Bmatrix}$$

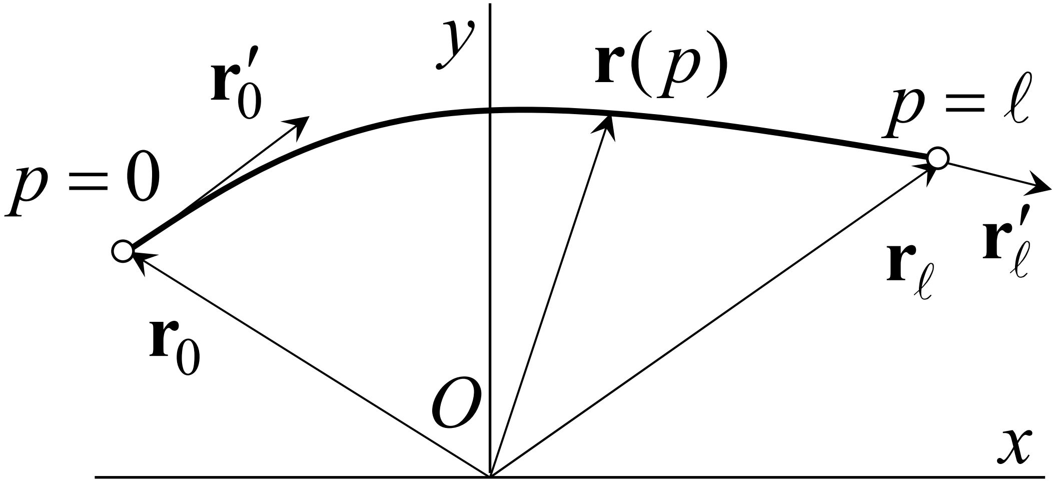
$$\mathbf{M}_{b1} = \begin{bmatrix} p & p \\ p & p \end{bmatrix}$$

$$\mathbf{q}_{b2} = \begin{Bmatrix} \mathbf{r}_4 \\ \boldsymbol{\varphi}_5 \end{Bmatrix}$$

$$\mathbf{M}_{b2} = \begin{bmatrix} p & p \\ p & p \end{bmatrix}$$

2. Absolute nodal coordinate formulation (ANCF)

2.1. ANCF 2D beam finite element by *A. Shabana*



Radius vector

$$\mathbf{r}(p, \mathbf{q}) = \mathbf{S}(p)\,\mathbf{q}$$

Equations of motion
(in the large displacement case)

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{Q}^{\text{elast}}(\mathbf{q}) = \mathbf{Q}^{\text{grav}}$$

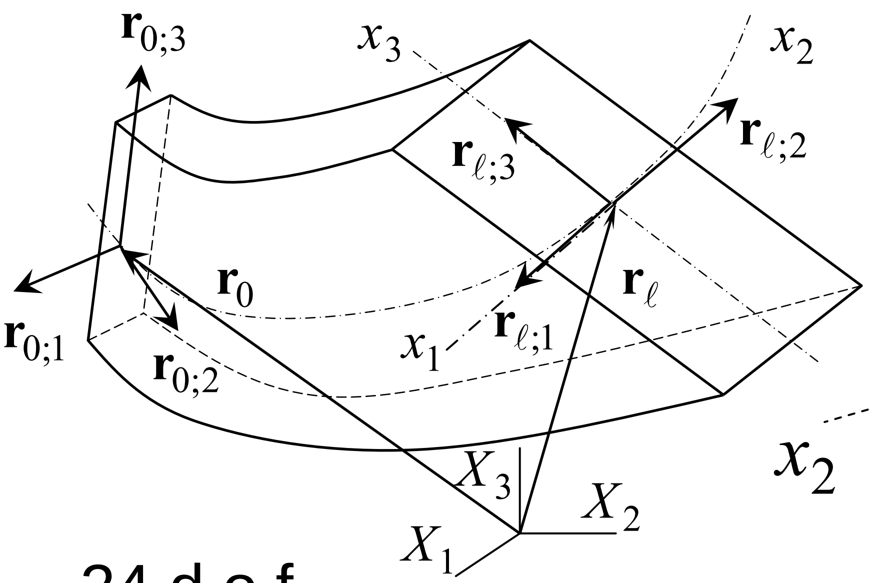
Constant mass matrix

The only nonlinear term

Constant gravity
applied forces

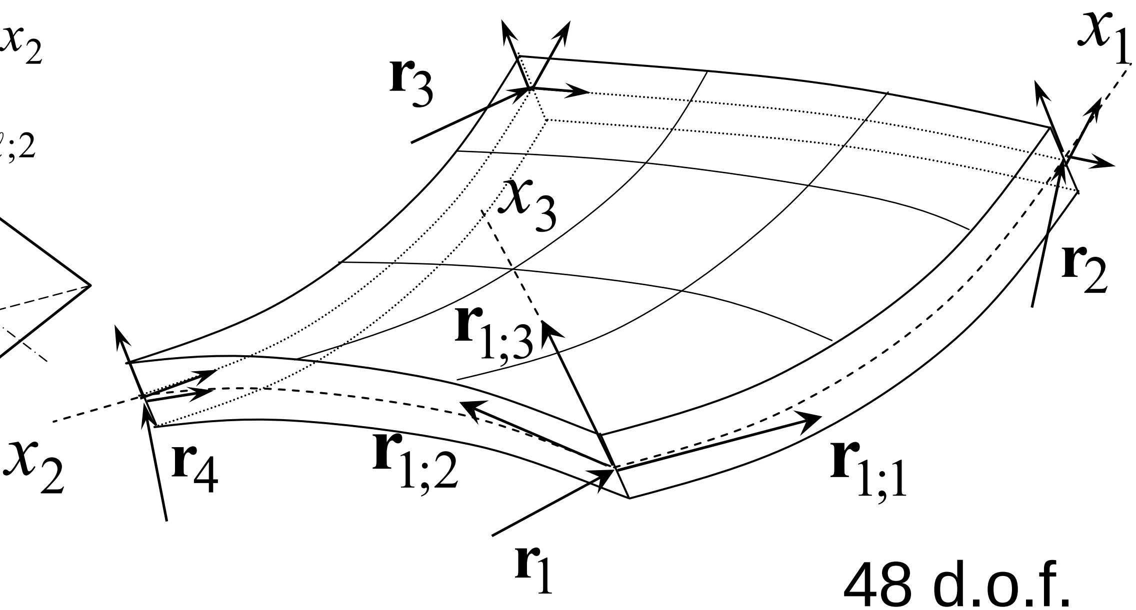
2.2. Thick beam/plate/shell elements by Shabana et al.

A. Shabana, R. Yakoub (2001)



24 d.o.f.

A. Mikkola, A. Shabana (2001)



48 d.o.f.

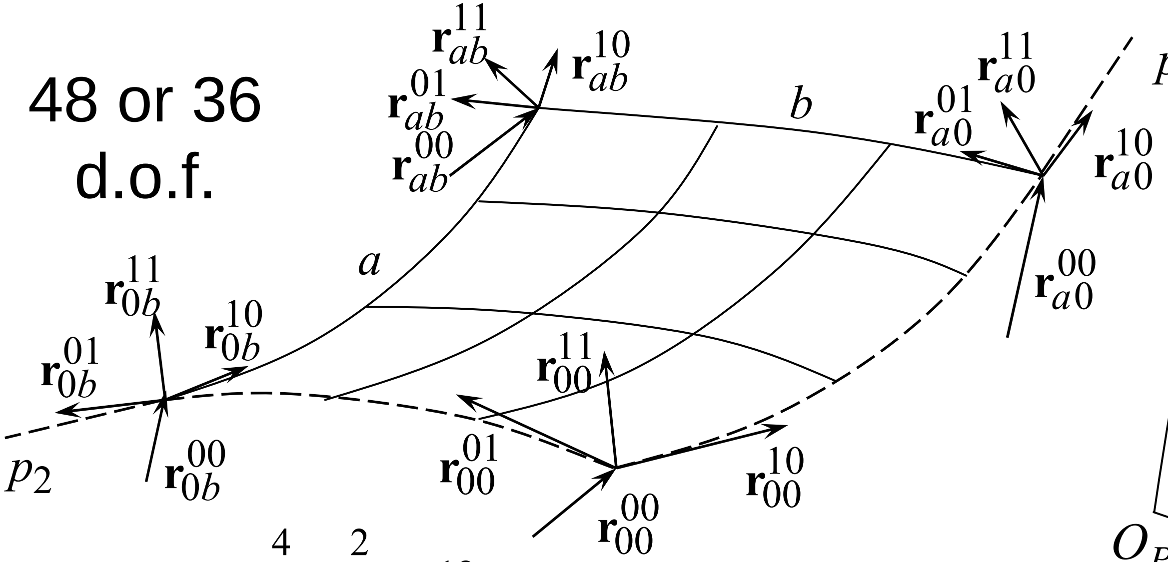
$$\mathbf{r}_{\alpha;k} = \frac{\partial \mathbf{r}_\alpha}{\partial x_k}$$

$$\mathbf{r} = \mathbf{S}(x_1, x_2, x_3)\,\mathbf{q}$$

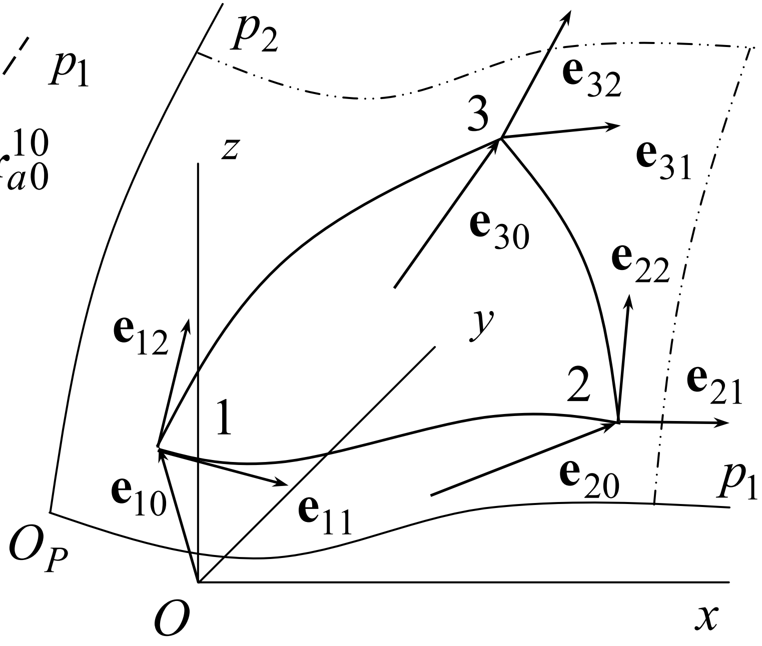
The elements represent **thick** beams / plates / shells

2.3. Thin elements by Dmitrochenko & Pogorelov (2002)

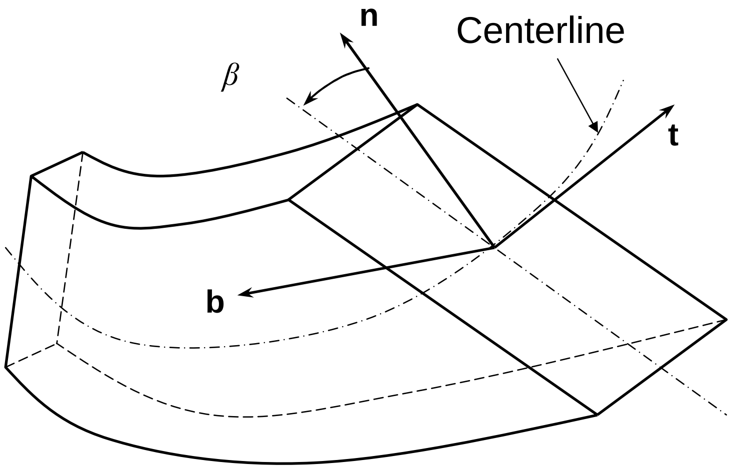
48 or 36
d.o.f.



$$\mathbf{r} = \sum_{m=1}^{4} \sum_{n=0}^{2} S_{mn}^{12}(p_1, p_2)\,\mathbf{e}_{mn}$$



27 d.o.f.



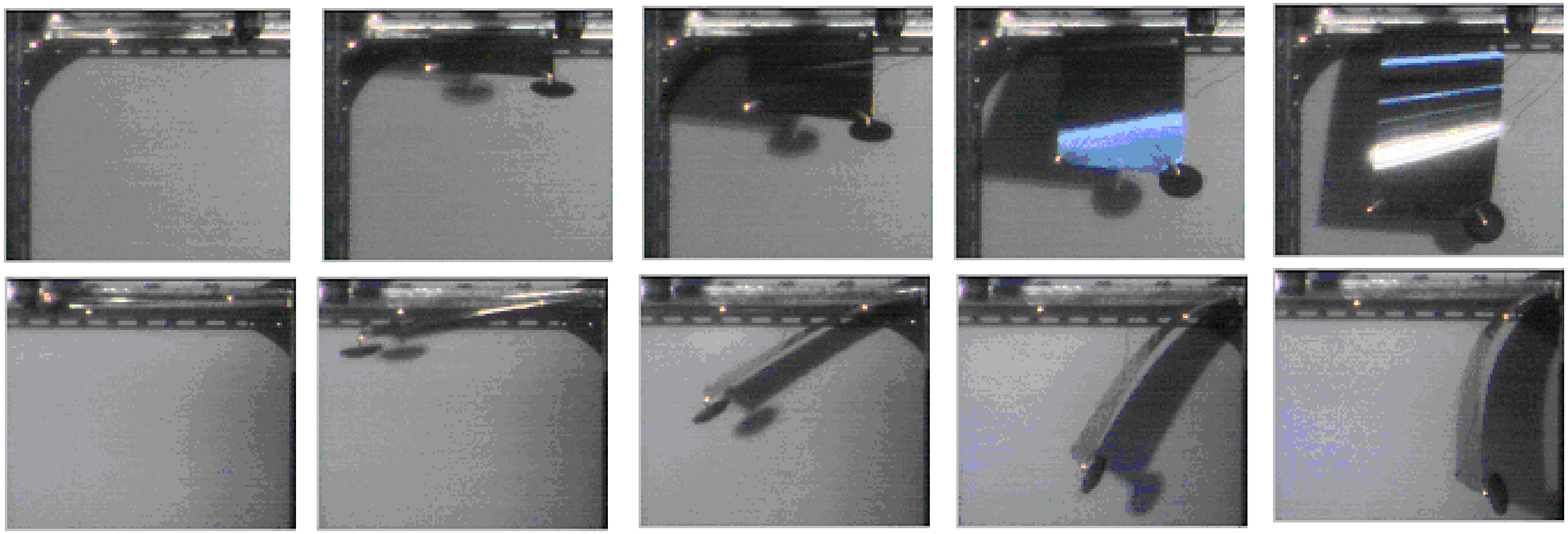
14 d.o.f.

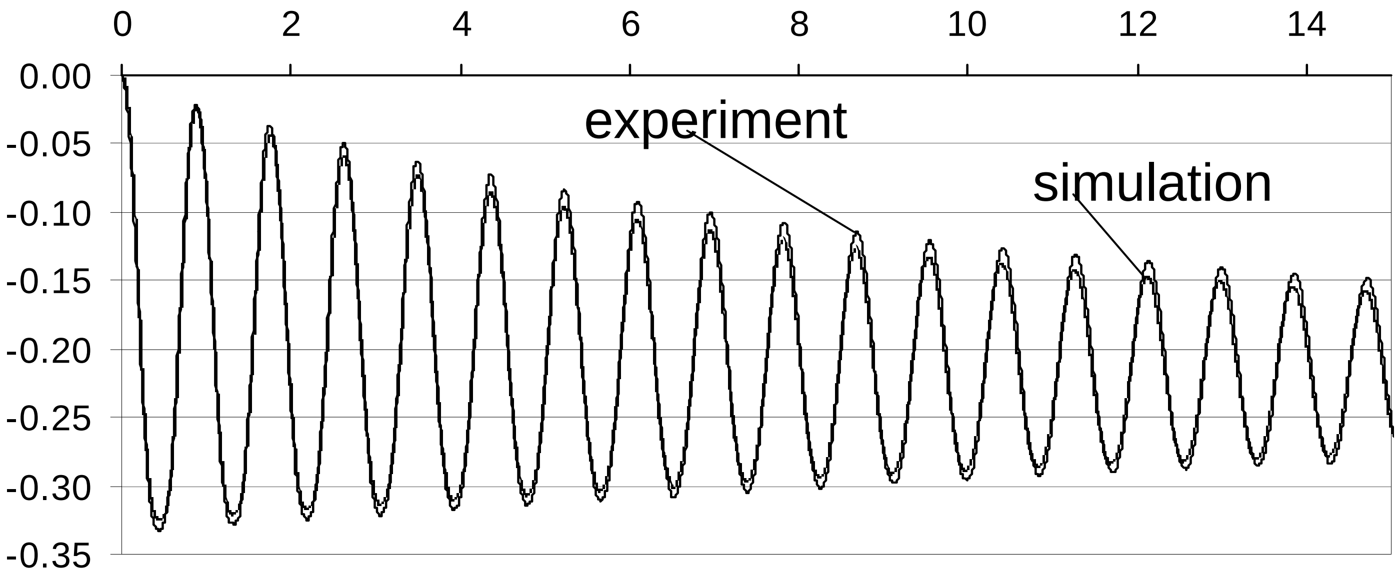
$$\mathbf{r} = \sum_{m=1}^{3} \sum_{n=0}^{2} S_{mn}^{L}(L_1, L_2, L_3)\,\mathbf{e}_{mn}$$

The elements represent **thin** beams and plates

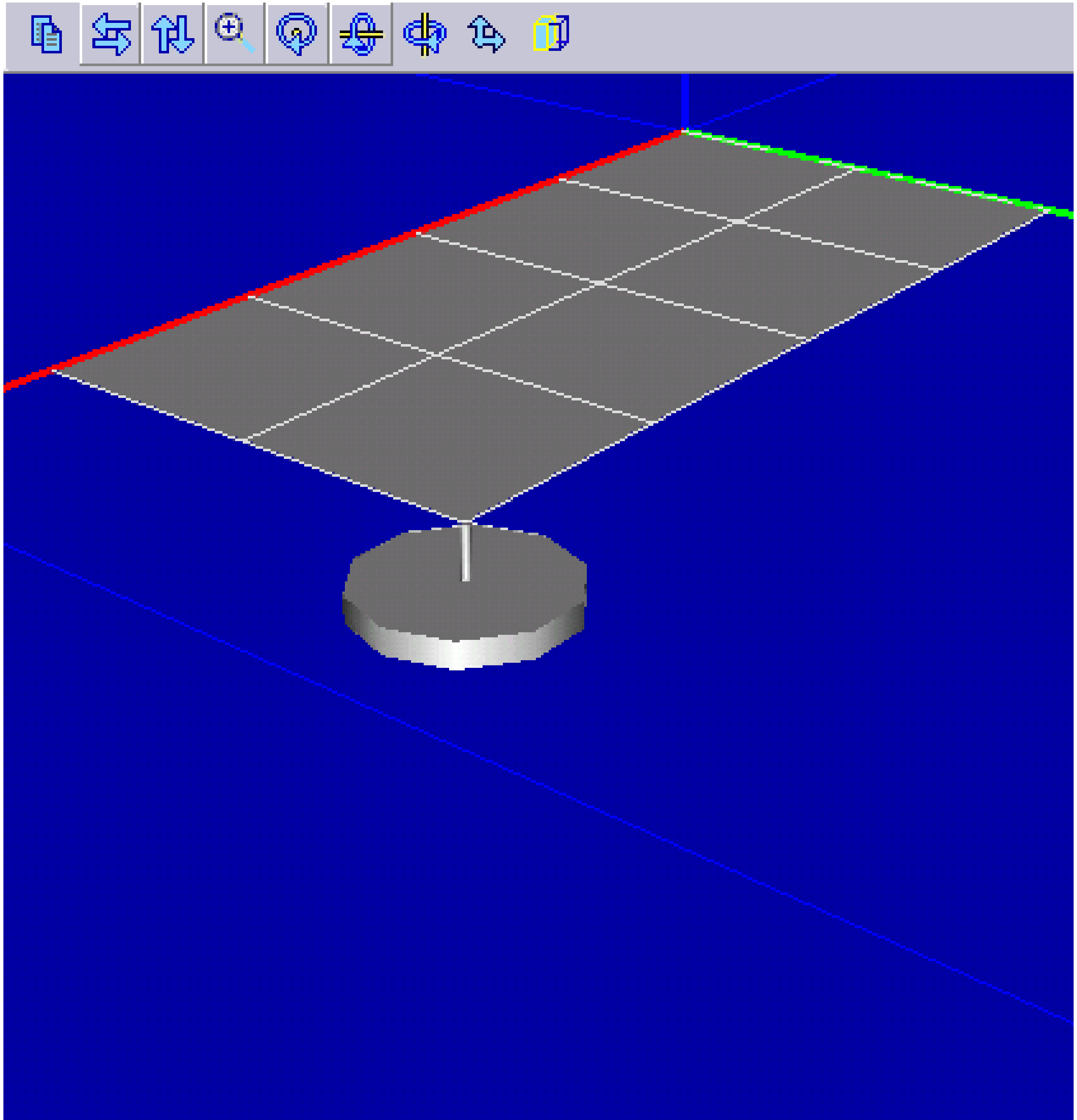
2.4. Examples: Cantilever beam, Clamped plate

Experiments were done at Pusan National University (Korea) jointly with Prof. Wan-Suk Yoo (CAE Lab)



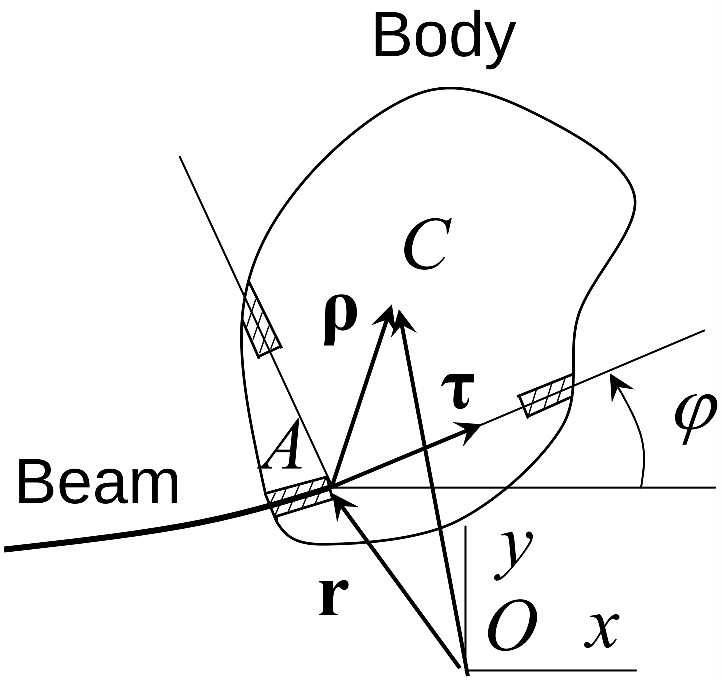


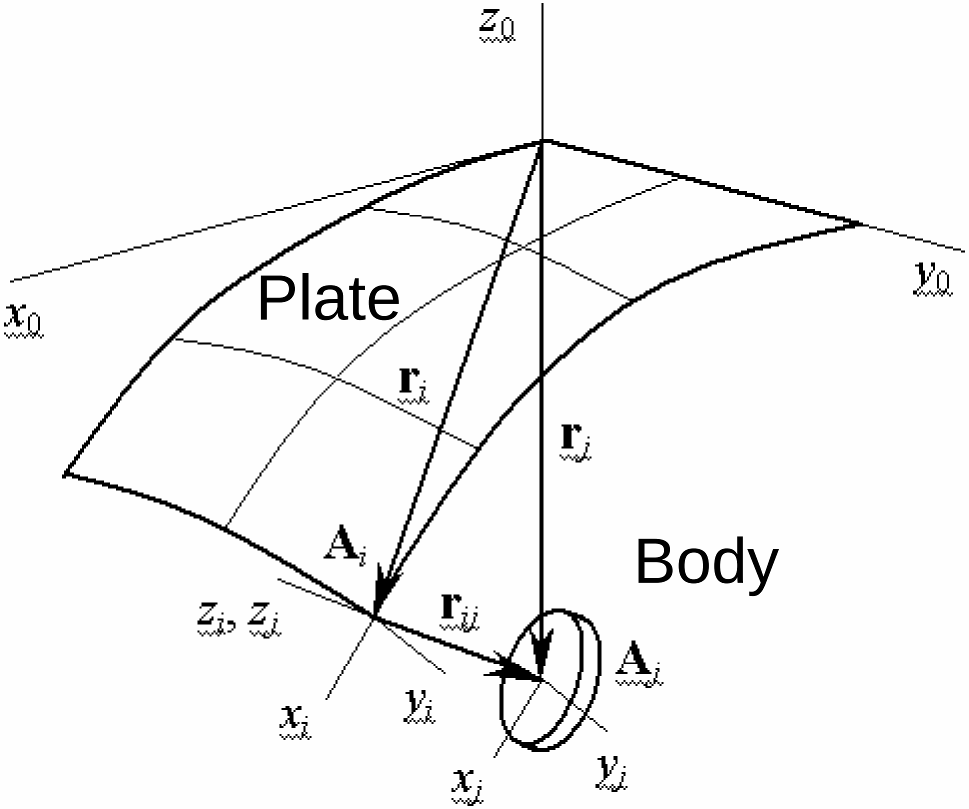
2.5. Animation of a clamped plate with a weight



Simulation tool is
Universal Mechanism
by Prof. *D. Pogorelov*

2.6. Simulation of beam/plate and weight using DAE

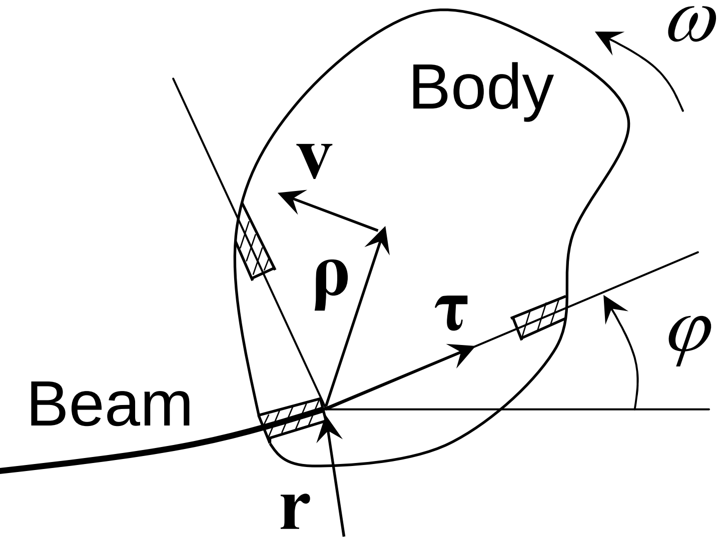


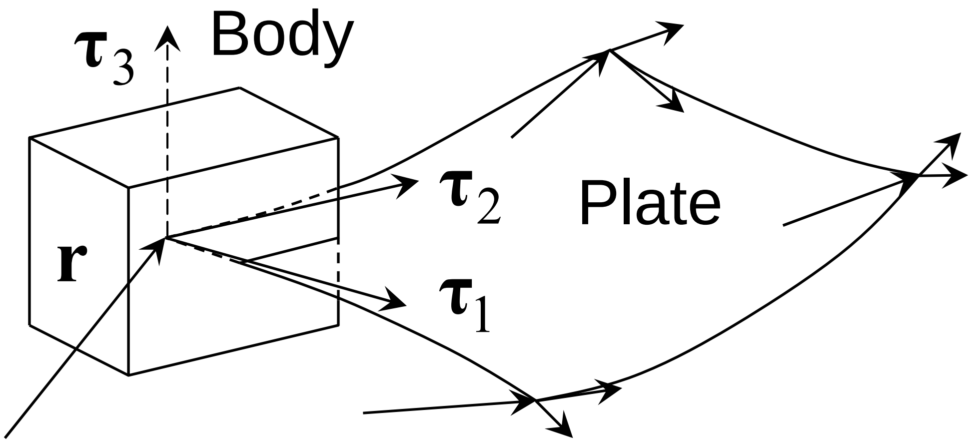


$$\begin{cases} \mathbf{M}_i\ddot{\mathbf{q}}_i = \mathbf{Q}_i + \mathbf{G}_i^T\boldsymbol{\lambda} \\ \mathbf{M}_j\ddot{\mathbf{q}}_j = \mathbf{Q}_j + \mathbf{G}_j^T\boldsymbol{\lambda} \\ \begin{Bmatrix} \mathbf{r} + \boldsymbol{\rho} - \mathbf{r}_C \\ \tan\varphi - (\tau_y/\tau_x) \end{Bmatrix} = \mathbf{0} \end{cases}$$

$$\begin{cases} \mathbf{M}_i\ddot{\mathbf{q}}_i = \mathbf{Q}_i + \mathbf{G}_i^T\boldsymbol{\lambda} \\ \mathbf{M}_j\ddot{\mathbf{q}}_j = \mathbf{Q}_j + \mathbf{G}_j^T\boldsymbol{\lambda} \\ \begin{Bmatrix} \mathbf{r}_i + \mathbf{r}_{ij} - \mathbf{r}_j \\ \sum_{k=1}^{3} \tilde{\boldsymbol{\iota}}_k \left[\mathbf{A}_i\mathbf{A}_{ij}\mathbf{A}_j^{-1}\right]\boldsymbol{\iota}_k \end{Bmatrix} = \mathbf{0} \end{cases}$$

2.7. Elimination of the algebraic part from DAE





$\mathbf{x} = \{\mathbf{r}^{\mathrm{T}}\ \boldsymbol{\tau}^{\mathrm{T}}\}^{\mathrm{T}}$ – Coordinates of body – $\mathbf{x} = \{\mathbf{r}^{\mathrm{T}}\ \boldsymbol{\tau}_1^{\mathrm{T}}\ \boldsymbol{\tau}_2^{\mathrm{T}}\}^{\mathrm{T}}$

$$\varphi = \mathrm{arctg}(\tau_Y / \tau_X)$$

$$\omega = \dot{\varphi}(\mathbf{x}, \dot{\mathbf{x}})$$

$$\mathbf{v} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

$$\Big\downarrow$$

$$\mathbf{v} = \boldsymbol{\Phi}\,\dot{\mathbf{x}}$$

$$\mathbf{a} = \boldsymbol{\Phi}\,\ddot{\mathbf{x}} + \dot{\boldsymbol{\Phi}}\,\dot{\mathbf{x}}$$

$$\mathbf{A} \approx [\boldsymbol{\tau}_1\ \boldsymbol{\tau}_2\ \boldsymbol{\tau}_3]$$

$$\tilde{\boldsymbol{\omega}} = \dot{\mathbf{A}}\,\mathbf{A}^T$$

Virtual work principle

Equations of motion (ODE)

$$\int_V \delta \mathbf{r}^T \mu (\mathbf{a} - \mathbf{g})\, \mathrm{d}V = 0 \longrightarrow \mathbf{M}(\mathbf{x})\,\ddot{\mathbf{x}} + \mathbf{Q}^{\mathrm{inert}}(\dot{\mathbf{x}}, \mathbf{x}) = \mathbf{Q}^{\mathrm{grav}}(\mathbf{x})$$

Conclusions

**Coupled rigid and elastic bodies
can be modeled as a constraint-free problem.**

We considered two finite-element large-displacement formulations to simulate elastic bodies:

1) **large rotation vector formulation** by J. Simo and
2) **absolute nodal coordinate formulation** by A. Shabana.

We considered two cases of attaching a rigid body to elastic one:

1) without restrictions for relative angular orientation (**revolute joint** in 2D case, **spherical joint** in 3D case)
2) without relative degrees of freedom (**the bodies are clamped**).

It can be shown that **this technique can be generalized** for any kind of joints between elastic and rigid bodies.

Thank you for your kind attention.